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Translated by L.K.

PMM U.S.S.R.,Vol.51,No.2,pp.196-201,1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

INVESTIGATION OF PARTIAL ASYMPTOTIC STABILITY AND INSTABILITY BASED ON THE LIMITING EQUATIONS*

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A new type of limiting equations is studied, used to investigate the asymptotic stability and instability of unperturbed motion with respect to some of the variables, based on the Lyapunov function with a sing-constant derivative, without assuming that the perturbed motions are bounded over the non-controlled coordinates. Sufficient conditions are derived for the asymptotic stability with respect to the generalized velocities and some of the generalized coordinates of the zero position of equilibrium of the non-autonomous, holonomic and non-holonomic mechanical systems under the action of dissipative forces.

1. Let us consider the following system of equations:

$$x^{\cdot} = X(t, x) \ (X(t, 0) \equiv 0) \tag{1.1}$$
$$x \in R^m, x = (y, z), \ y \in R^s, \ z \in R^p \ (m = s + p)$$

The function $X(t, x)$: $R^+ \times \Gamma \to R^m$ ($R^+ = [0, +\infty[$, $\Gamma = \{\|y\| < H > 0, \ \|z\| < +\infty\}$, $\|y\|$ is a norm in R^s, $\|z\|$ in R^p, $\|x\| = \|y\| + \|z\|$) satisfies the conditions for the existence of solutions in the Caratheodory sense /1/. A locally integrable function $r(t) \in L_1$ exists, continuous in x for fixed t, measurable in t for fixed x, for every compact set $K \subset \Gamma$ such that $\|X(t, x)\| \leqslant r(t)$. We shall also assume that system (1.1) satisfies the conditions of z-continuability of the solutions /2/.

We will also introduce a shift of the function $X(t, x)$ in t by an amount $\tau \geqslant 0$ according

*Prikl.Matem.Mekhan.,51,2,253-259,1987

to the formula $X_\tau(t, x) = X(t+\tau, x)$, a shift in t and z by an amount $\tau \geqslant 0$, and a vector $\eta \in R^p$ according to the formula $X_{\tau\eta}(t, x) = X(t+\tau, y, z+\eta)$.

Definition 1. System (1.1) will be called (t, z)-precompact, if for any sequences $\tau_n \to +\infty$ and $\eta_n \to \infty$ there exist subsequences $\{\tau_{nk}\}$ and $\{\eta_{nk}\}$ and limiting functions $\Phi_1(t, x)$: $R^+ \times \Gamma \to R^m$ and $\Phi_2(t, x)$: $R^+ \times \Gamma \to R^m$, such, that the following relations hold for any sequence of continuous functions $u = u^*(t)$: $[a, b] \to \Gamma$, converging uniformly to the function $u_k(t)$: $[a, b] \to \Gamma$:

$$\int_a^b \Phi_1(t, u^*(t))\,dt = \lim_{k\to+\infty} \int_a^b X_\tau^{(k)}(t, u_k(t))\,dt \tag{1.2}$$

$$\int_a^b \Phi_2(t, u^*(t))\,dt = \lim_{k\to+\infty} \int_a^b X_{\tau\eta}^{(k)}(t, u_k(t))\,dt$$

$$X_\tau^{(k)}(t, x) = X(t+\tau_{nk}, x),\ X_{\tau\eta}^{(k)}(t, x) = X(t+\tau_{nk}, y, z+\eta_{nk})$$

Moreover, we shall call the systems of equations

$$x^\cdot = \Phi_1(t, x),\ x^\cdot = \Phi_2(t, x) \tag{1.3}$$

t- and (t, z)-limiting with respect to (1.1) respectively.

The precompactibility of (1.1) and a relation connecting the solutions of (1.1) and (1.3), will be established with help of the following lemmas obtained in the same manner as those in /3, 4/.

Lemma 1. The sufficient condition for system (1.1) to be precompact is, that the function $X(t, x)$ satisfy the following conditions: for every set $S = \{\|y\| \leqslant H_1 < H,\ \|z\| < +\infty\}$ there exist locally two functions $\nu = \nu(S, t) \in L_1$ and $\vartheta = \vartheta(S, t) \in L_1$, so that

$$\|X(t, x)\| \leqslant \nu(S, t), \tag{1.4}$$

$$\|X(t, x_2) - X(t, x_1)\| \leqslant \vartheta(S, t)\|x_2 - x_1\|$$

Here the function $\nu(S, t)$ is uniformly continuous in the mean in the interval $[t, t+1]$, and the function $\vartheta(S, t)$ is bounded in the norm in the interval $[t, t+1]$, i.e.

$$\int_E \nu(S, t)\,dt < \varepsilon,\ \int_t^{t+1} \vartheta(S, t)\,dt \leqslant N$$

for any $\varepsilon > 0$ and $t \geqslant 0$ of any set $E \subset [t, t+1]$ with the measure $m(E) \leqslant \delta(\varepsilon) > 0$ and for some number $N = N(S)$. Under these conditions the solutions of systems (1.1) and (1.3) are also unique.

Lemma 2. We shall assume that system (1.1) satisfies conditions (1.4), and for some sequences $\tau_n \to +\infty$ and $\eta_n \to \infty$ the sequence $X_\tau^{(n)}(t, x) = X(t+\tau_n, x) \to \Phi_1(t, x)$ and $X_{\tau\eta}^{(n)}(t, x) = X(t+\tau_n, y, z+\eta_n) \to \Phi_2(t, x)$, and the sequence $x_n = (y_d, z_n) \to x_0 = (y_0, z_0)$. Let $\varphi_1(t)$: $[0, \alpha_1[\to \Gamma$ and $\varphi_2(t)$: $[0, \alpha_2[\to \Gamma$ $(\varphi_1(0) = \varphi_2(0) = x_0)$ be the solutions of the corresponding first and second system of (1.3). Then the sequence of solutions $x = x_n(t)$ of the system $x^\cdot = X(t+\tau_n, x)$, satisfying the initial conditions $x_n(0) = x_n$ converge to $x = \varphi_1(t)$ uniformly in every interval $[0, \beta_1] \subset [0, \alpha_1[$. The sequence of solutions $x' = x_n'(t)$ of the systems $x^\cdot = X(t+\tau_n, y, z+\eta_n)$ with initial conditions $x_n'(0) = x_n$ converges to $x = \varphi_2(t)$ uniformly in $[0, \beta_2] \subset [0, \alpha_2[$.

Note. Let us consider separately the case when system (1.1) is autonomous

$$x^\cdot = X(x) \tag{1.5}$$

The system will be z-precompact if for any sequence $\eta_n \to \infty$ there exists a subsequence $\{\eta_{nk}\} \subset \{\eta_n\}$ and a function $\Phi(x)$: $\Gamma \to R^m$ such, that $X(y, z+\eta_{nk}) \to \Phi(y, z)$ uniformly on every compact $\{\|y\| \leqslant H_1 < H, \|z\| \leqslant Q\}$. The system $x^\cdot = \Phi(x)$ will be z-limiting with respect to (1.5). The necessary and sufficient condition for (1.5) to be z-precompact is that the function $X(x)$ be bounded and uniformly continuous on every set $S = \{\|y\| \leqslant H_1 < H, \|z\| < +\infty\}$. The sufficient condition for the solutions of the system (1.5) and the systems limiting with respect to it is, that the function $X(x)$ satisfy the Lipshitz condition $\|X(x_2) - X(x_1)\| \leqslant L(\|x_2 - x_1\|)$.

It was suggested in /5, 6/ that the y-behaviour of the solutions of (1.1) can be determined by constructing the systems $y^\cdot = \Psi(t, y)$ limiting with respect to it, for every continuous function $z = z(t) \to \infty$. Introducing the (t, z)-limiting systems enables us, as compared with /5, 6/, to take into account the z-properties of the solutions of (1.1).

Let us denote by $W(t, x)$: $R^+ \times \Gamma \to R^+$ the function satisfying conditions of the form (1.4), which guarantee the precompactness of its shifts $W_\tau^{(n)}(t, x) = W(t+\tau_n, x)$ and $W_{\tau\eta}^{(n)}(t, y, z) = W(t+\tau_n, y, z+\eta_n)$ in the convergence (1.2).

Definition 2. The function $\omega(t,x) = (\omega_1(t,x),\ \omega_2(t,x))$: $R^+ \times \Gamma \to R^+ \times R^+$ will be called (t,z)-limiting with respect to $W(t,x)$, if sequences $\tau_n \to +\infty$ and $\eta_n \in R^p$, $\eta_n \to \infty$, exist such that the sequences of shifts $W_\tau^{(n)}(t,x) = W(t+\tau_n, x)$ and $W_{\tau\eta}^{(n)}(t,y,z) = W(t+\tau_n, y, z+\eta_n)$ converge in the convergence (1.2) to $\omega_1(t,x)$ and $\omega_2(t,x)$ respectively.

Definition 3. The limiting functions $\Phi = (\Phi_1, \Phi_2)$ and $\omega = (\omega_1, \omega_2)$ form a limiting pair (Φ, ω), provided that they are limiting for the same sequences $\tau_n \to +\infty$ and $\eta_n \to \infty$.

Definition 4. Let V: $R^+ \times \Gamma \to R^+$ be a Lyapunov function, and $\tau_n \to +\infty$ and $\eta_n \to \infty$ be some sequences. We will define the limiting sets $N_1(t,c)$ and $N_2(t,c)$ as the sets of points x such, that $x \in N_1(t,c)$, provided that a sequence $x_n \to x$ exists for which $V(t+\tau_n, x_n) \to c$ as $n \to +\infty$, $x \in N_2(t,c)$, if a sequence $x_n = (y, z_n) \to x$ exists for which $V(t+\tau_n, y_n, z_n+\eta_n) \to c$ as $n \to +\infty$.

2. Let V: $R^+ \times \Gamma \to R^+$ be a Lyapunov function possessing, by virtue of (1.1): $V^\cdot(t,x) \leqslant -W(t,x) \leqslant 0$, a derivative. Let (Φ, ω) be a limiting pair and $N(t,c) = N_1(t,c)\ N_2(t,c)$ a corresponding limiting set. We denote by M_i $(i = 1, 2)$ the set of solutions of the corresponding system (1.3), belonging in its whole interval of definition to the set $\{N_i(t,c)$: $c = c_0 = \mathrm{const} \geqslant 0\} \cap \{\omega_i(t,x) = 0\}$, $\bigcup M_1$, and $\bigcup M_2$ are the unions of the sets M_1 and M_2 over all limiting pairs (Φ, ω). We shall write $M_y^+ = (\bigcup M_1)_y \cup (\bigcup M_2)_y$ where $(\ )_y$ is the projection of the set on the hyperplane $z = 0$.

Theorem 1. We shall assume that a function $V(t,x) \geqslant 0$ exists, with a derivative $V^\cdot(t,x) \leqslant -W(t,x) \leqslant 0$. Then every solution of (1.1) $x = x(t, t_0, x_0)$ bounded in y so that $\|y(t,t_0,x_0)\| \leqslant H_1 < H$ for all $t \geqslant t_0$, approaches M_y^+ in y without limit as $t \to +\infty$, i.e. $y(t,t_0,x_0) \to M_y^+$ as $t \to +\infty$.

Proof. Let $x = x(t,t_0,x_0)$ be a solution of system (1.1) bounded in y, $\|y(t,t_0,x_0)\| \leqslant H_1$ at all $t \geqslant t_0$.

The function $V(t) = V(t, x(t,t_0,x_0))$ decreases monotonically and has a lower limit. This means that $c_0 \geqslant 0$ exists such, that $V(t) \to c_0$ as $t \to +\infty$, or

$$\lim_{n \to +\infty} V(t_n + t,\ x(t_n + t,\ t_0,\ x_0)) = c_0 \tag{2.1}$$

for any sequence $t_n \to +\infty$ and any $t \geqslant 0$.

Let y_0^* be a y-limiting point of this solution /7/, i.e. there exists $\tau_n \to +\infty$ such that $y(\tau_n, t_0, x_0) \to y_0^*$. Two cases are possible: the sequence $z_n = z(\tau_n, t_0, x_0)$ is bounded, and $\eta_n = z(\tau_n, t_0, x_0) \to \infty$.

In the first case we can assume that $z_n \to z_0^*$. This means that $x_n = x(\tau_n, t_0, x_0) \to x_0^* = (y_0^*, z_0^*)$. Let us assume that $x_n(t) = x(\tau_n + t,\ t_0, x_0)$. Repeating the arguments used in Theorem 2.1 of /8/ and taking into account (2.1), we can establish that a subsequence $n_k \to +\infty$ exists for which $x_{nk}(t)$ will converge to the non-continuable solution $x = \varphi_1(t)$ of some limiting system $x^\cdot = \Phi_1^*(t,x)$, such that $\varphi_1(t) \in N_1^*(t, c_0) \cap \{\omega_1^*(t,x) = 0\}$. From this it follows that $y_0^* \in (M_1^*)_y \in M_y^+$.

In the second case we will assume that $x_n'(t) = (y_n'(t), z'(t)) = (y_n(t), z_n(t) - \eta_n)$. The functions $x_n'(t)$ will be the solutions of systems $x^\cdot = X_{\tau\eta}^{(n)}(t,x)$, satisfying the initial conditions $x_n'(0) = (y_n, 0)$. The following estimates follow from the inequality $V^\cdot(t,x) \leqslant -W(t,x)$:

$$V(\tau_n + t) - V(\tau_n) \leqslant -\int_0^t W_{\tau\eta}^{(n)}(s, y_n'(s), z_n'(s))\, ds$$

$$W_{\tau\eta}^{(n)}(t,x) = W(\tau_n + t, y, \eta_n + z)$$

Continuing the further arguments as in the case of bounded η_n, we conclude that there exists a non-continuable solution $x = \varphi_2(t)$ of some limiting system $x^\cdot = \Phi_2^*(t,x)$, satisfying the initial condition $\varphi_2(0) = (y_0^*, 0)$, and such, that $\varphi_2(t) \in N_2^*(t, c_0) \cap \{\omega_2^*(t,x) = 0\}$. This implies that $(y_0^*, 0) \in \varphi_2(t) \subset M_2^*$, i.e. $y_0^* \in (\bigcup M_2)_y \subset M_2^+$. The theorem is proved.

Theorem 2. We shall assume that 1) there exists a y-positive-definite function $V(t,x)$ with a derivative by virtue of (1.1) $V^\cdot(t,x) \leqslant -W(t,x) \leqslant 0$; 2) every limiting pair (Φ, ω) with the set $N(t,c)$ has the following property: solutions of every system (1.3) belonging, respectively, to the sets $\{N_i(t,c)$: $c = \mathrm{const}\} \cap \{\omega_i(t,x) = 0\}$ $(i = 1, 2)$, belong also to the set $\{x : y = 0\}$. Then the zero solution of (1.1) is asymtotically y-stable.

Its proof follows from Theorem 1.

Let the function $V = V(t,x)$ be bounded and satisfy the Lipshitz condition in t and x in every region $\{t \geqslant 0, \|y\| \leqslant H_1 < H, \|z\| < +\infty\}$. Then the function will have an infinitesimal upper limit in x, for any sequences $\tau_n \to +\infty$ and $\eta_n \to \infty$ there exist subsequences $\{\tau_{nk}\} \subset \{\tau_n\}$ and $\{\eta_{nk}\} \subset \{\eta_n\}$, such, that the sequences of functions $V_\tau^{(k)}(t,x) = V(t + \tau_{nk}, x)$ and

$V_{\tau\eta}^{(k)}(t, x) = V(t + \tau_{nk}, y, z + \eta_{nk})$ converge to some functions $\rho_1(t, x)$ and $\rho_2(t, x)$ uniformly on every compact set $[0, T] \times \{\|y\| \leqslant H_1 < H, \|z\| \leqslant Q\}$.

Let us determine the limiting set $(\Phi, \rho, \omega) = ((\Phi_1, \Phi_2), (\rho_1, \rho_2), (\omega_1, \omega_2))$ as a set in which the functions Φ_1, ρ_1 and ω_1, Φ_2, ρ_2 and ω_2 are limiting functions for the same sequences $\tau_n \to +\infty$ and $\eta_n \to \infty$.

Theorem 3. Let us assume that 1) there exists a bounded y-positive definite function $V(t, x)$, satisfying the Lipshitz condition in t and x with a derivative $V^{\cdot}(t, x) \leqslant -W(t, x) \leqslant 0$; 2) , for every limiting set (Φ, ρ, ω) the sets $\{\rho_i(t, x) = \mathrm{const} > 0\} \cap \{\omega_i(t, x) = 0\}$ $(i = 1, 2)$ do not contain any solutions of the corresponding system (1.3). Then the zero solution of (1.1) is uniformly asymptotically y-stable.

The proof follows that of Theorem 2.4 of /9/. First we show that every limiting set (Φ, ρ, ω) has the following property: if $\varphi(t, t_0, x_0)$ is a solution of the system $x^{\cdot} = \Phi(t, x)$, then $\rho(t, \varphi(t, t_0, x_0)) \to 0$ as $t \to +\infty$. Next, using reductio ad absurdum we show that the function $V(t, x(t, t_0, x_0)) \to 0$ as $t \to +\infty$ uniformly in t_0 and x_0 along the solutions of (1.1) bounded in y by the region $\{x: \|y\| \leqslant H_1 < H\}$.

Modifying the proofs of the above theorems we can obtain the following additional results.

Theorem 4. We will assume that 1) there exists a y-positive-definite function $V(t, x)$, whose derivative $V^{\cdot}(t, x) \leqslant -W(t, x) \leqslant 0$; 2) there exists a sequence $t_n \to +\infty$, for which every limiting pair (Φ, ω) with the set $N(t, c)$ will be such, that the sets $\{N_i(t, c)\colon c = \mathrm{const} > 0\} \cap \{\omega_i(t, x) = 0\}$ $(i = 1, 2)$ will contain no solutions of the corresponding systems (1.3). Then the zero solution of the system (1.1) will be asymptotically y-stable uniformly in x_0.

Theorem 5. We will assume that a function $V(t, x)$, exists which in any small neighbourhood of $x = 0$ takes positive values, is bounded in the region $V(t, x) \geqslant 0$ and possesses a derivative $V^{\cdot}(t, x) \geqslant W(t, x) \geqslant 0$. Condition 2) of Theorem 4 also holds. Then the zero solution of system (1.1) will be unstable in y.

The theorems obtained extend and generalize the results of /10-14, 5, 6, 9/. Unlike in /10-12/, the condition of positive negativeness of the derivative in some of the variables is replaced by the condition of its constant negativeness. At the same time, there is no demand of z-boundedness which appears in /9, 13, 14/, and unlike in /5, 6/, we take into account the z-properties of the system as $z \to \infty$.

Examples. 1°. Let us consider the autonomous system of equations

$$y^{\cdot} = -y\cos^2 z,\quad z^{\cdot} = f_1(y)\sin^2 z \tag{2.2}$$

where $f_1(y) \neq 0$ when $y \neq 0$. By virtue of (2.2) we have $V^{\cdot} = -2y^2\cos^2 z$ for any function $V = y^2$. The systems z-limiting with respect to (2.2) will have the form (2.2) when z is replaced by $z + \gamma$ ($\gamma = \mathrm{const}$). The corresponding functions of $\omega(y, z) = -2y^2\cos^2(z + \gamma)$, z-limiting with respect to $V^{\cdot} = -2y^2\cos^2 z$, which are z-limiting with respect to $V = y^2$, are identical with V. But the set $\{V = \mathrm{const} > 0\} \cap \{\omega = 0\} = \{y \neq 0, \cos^2(z + \gamma) = 0\}$ contains no solutions of the z-limiting system. Therefore, according to Theorem 3 the zero solution of the system (2.2) is uniformly asymptotically stable in y.

2°. A problem of asymptotic stability with respect to some of the variables and coordinates of the zero position of equilibrium of a holonomic mechanical system with Lagrange's function $L = L_2(t, q, q^{\cdot}) + L_1(q, q^{\cdot}) + L_0(t, q)$, acted upon by gyroscopic and dissipative forces $Q = Q(t, q, q^{\cdot})$, was discussed in /9/. Theorems 2-4 enable us to assert that Theorems 3.1 and 3.2 of /9/ remain valid, provided that the condition that the motions in $q_{m+1}, q_{m+2}, \ldots, q_n$ are bounded, is replaced by the condition that the right-hand sides of the equations of motion solved for $q^{\cdot\cdot}$ are bounded and satisfy the Lipshitz conditions in $t, q, q^{\cdot}$ in every region $\{q_1^{\cdot 2} + q_2^{\cdot 2} + \ldots + q_n^{\cdot 2} \leqslant H = \mathrm{const}, q_1^2 + q_2^2 + \ldots + q_m^2 \leqslant H, q_{m+1}^2 + q_{m+2}^2 + \ldots + q_n^2 < +\infty\}$.

3°. Let us consider the motion of a heavy material point along the surface $z = f(x, y)$. We shall assume that the function $f(x, y)$ is positive definite in x, the function and its partial derivatives up to and including the second order are bounded and equicontinuous in the region $\{\|x\| \leqslant H_1, \|y\| < +\infty\}$; $\partial f/\partial x = \partial f/\partial y = 0$ when $x = y = 0$, so that the point has a position of equilibrium $x^{\cdot} = y^{\cdot} = x = y = 0$. Let the point be also acted upon by dissipative forces Q_x and Q_y, bounded and equicontinuous in $t, x^{\cdot}, y^{\cdot}, x, y$. For the derivative of total energy $H = T + gf(x, y)$, positive-definite in $x^{\cdot}, y^{\cdot}, x$, we have $H^{\cdot} = Q_x x^{\cdot} + Q_y y^{\cdot}$. The equation in $x^{\cdot\cdot}$ will have the form

$$x^{\cdot\cdot} = \Bigl(-\frac{\partial f}{\partial x}\Bigl(g + \frac{\partial^2 f}{\partial x^2}x^{\cdot 2} + 2\frac{\partial^2 f}{\partial x\partial y}x^{\cdot}y^{\cdot} + \frac{\partial^2 f}{\partial y^2}y^{\cdot 2}\Bigr) + Q_x\Bigl(1 + \frac{\partial f}{\partial x}\Bigr)^2 + Q_y\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\Bigr)\Big/\Bigl(1 + \Bigl(\frac{\partial f}{\partial x}\Bigr)^2 + \Bigl(\frac{\partial f}{\partial y}\Bigr)^2\Bigr) \tag{2.3}$$

The equations which are t-, (t, y)-continuous with respect to (2.3) have the same form, but with the function $f^*(x, y)$ and values Q_x^* and Q_y^*, which are, respectively, limiting with respect to f, Q_x, Q_y. We can find from the form of (2.3), that when $Q_x^* = Q_y^* = 0$, and $x^{\cdot}$ and $y^{\cdot}$ are small, the solutions of the limiting equations satisfying the condition $x^{\cdot} \equiv 0$, must also satisfy the relation $\partial f^*/\partial x = 0$. This yields, according to Theorems 2 and 3, the following results.

If $|\partial f/\partial x| \geqslant h_1(|x|) \geqslant 0$, $Q_x x^\cdot + Q_y y^\cdot \leqslant -\beta(t) h_2(|x^\cdot|)$, where $\beta(t) \geqslant 0$, $\beta(t) \geqslant \beta_0 > 0$ when $t \in [t_n, t_n + \nu]$ $(t_n \to +\infty,\ t_{n+1} - t_n \leqslant r = \mathrm{const},\ \nu > 0)$ $h(a) = 0 \leftrightarrow a = 0$, then the zero position of equilibrium of the point stable in $x^\cdot, y^\cdot$ and x is asymptotically stable in $x^\cdot$ and x.

If for any small $\varepsilon > 0$ we have the inequality $|\partial f/\partial x| \geqslant \varepsilon > 0$ when $f(x, y) \geqslant \delta = \delta(\varepsilon) > 0$, and also $Q_x x^\cdot + Q_y y^\cdot \leqslant \beta(t) h(|x^\cdot| + |y^\cdot|)$, then the zero position of equilibrium of the point is uniformly asymptotically stable in $x^\cdot, y^\cdot, x$.

3. Let us consider the motion of a mechanical system with generalized coordinates $q_1, q_2, \ldots, q_m$, power-dependent on the linear, non-holonomic stationary constraints, and acted upon by potential, gyroscopic and dissipative forces depending explicitly on time. We take, as the equations of motion, the equations in the Voronets form /15/

$$\begin{gathered} q_2^\cdot = B(q)\, q_1^\cdot \\ \frac{d}{dt}\left(\frac{\partial \Theta}{\partial q_1^\cdot}\right) - \frac{\partial}{\partial q_1}(\Theta - \Pi) - \frac{\partial}{\partial q_2}(\Theta - \Pi) B = Q_1 + G_1 q_1^\cdot \\ q = (q_1, q_2),\quad q_1 \in R^s,\quad q_2 \in R^p \quad (s + p = m) \end{gathered} \tag{3.1}$$

where $B = B(q)$ is the matrix of the coefficients of the non-holonomic constraints, $2\Theta = (q_1^\cdot)^T A(q)\, q_1^\cdot$ is the reduced kinetic energy of the system with the constraints accounted for (A is an $s \times s$ matrix), $\Pi = \Pi(t, q)$ is the potential energy, $Q_1 = Q_1(t, q, q_1^\cdot)$ is the resultant of the generalized gyroscopic and dissipative forces, $G_1 q_1^\cdot$ are the non-holonomic, gyroscopic-type terms.

Let us assume that $\partial \Pi/\partial q = 0$ when $q = 0$. Then system (3.1) will have a zero position of equilibrium

$$q^\cdot = q = 0 \tag{3.2}$$

The problem of the asymptotic stability of the position of equilibrium (3.2) of system (3.1) in $q^\cdot$ and q_1 was solved in /13, 16, 17/ under the assumption that the generalized forces do not depend **explicitly on time**, the solutions of system (3.1) belonging to some neighbourhood of (3.2) are bounded in q_2, and the points of the set $\{q_1 = 0,\ q_2 = \mathrm{const}\}$ represent the positions of equilibrium of (3.1). The problem was solved in /6/ under the assumption that $\Pi = \Pi(q)$ and the influences of the time t and coordinates q_2 disappear as $t \to +\infty$ and $q_2 \to \infty$. Theorem 2 enables us to **solve** the problem under more general assumptions.

Let us assume that $\Pi(t, 0) \equiv 0$, $\partial \Pi/\partial t \leqslant 0$. Then we have the following expression for the derivative of $H = \Theta + \Pi$:

$$H^\cdot = \frac{\partial \Pi}{\partial t} + Q_1^T q_1^\cdot \leqslant Q_1^T q_1^\cdot$$

Eqs.(3.1) solved for $q_1^{\cdot\cdot}$, will be:

$$\begin{gathered} q_2^\cdot = B q_1^\cdot \\ q_1^{\cdot\cdot} = \{(q_1^\cdot)^T C q_1^\cdot\} - A^{-1}\left(\frac{\partial \Pi}{\partial q_1} + \frac{\partial \Pi}{\partial q_2} B + Q_1 + G_1 q_1^\cdot\right) \end{gathered} \tag{3.3}$$

where $\{(q_1^\cdot)^T C q_1^\cdot\}$ is a set of forms quadratic in $q_1^\cdot$. Let us assume that the quantities $\Pi(t, q)$, $B(q)$, $\{C(q)\}$, $A(q)$, $\partial \Pi/\partial q$, $Q_1(t, q, q_1^\cdot)$, $G_1(t, q, q_1^\cdot)$ are bounded satisfy the Lipshitz conditions over all their variables in the region $\{\|q_1^\cdot\| \leqslant H_1,\ \|q_1\| \leqslant H_1,\ \|q_2\| < +\infty\}$. Then the equations which are limiting with respect to (3.3) have the same form

$$\begin{gathered} q_2^\cdot = B^* q_1^\cdot \\ q_1^{\cdot\cdot} = \{(q_1^\cdot)^T C^* q_1^\cdot\} - (A^*)^{-1}\left(\left(\frac{\partial \Pi}{\partial q_1} + \frac{\partial \Pi}{\partial q_2} B\right)^* + Q_1^* + G_1^* q_1^\cdot\right) \end{gathered} \tag{3.4}$$

where the asterisk denotes an expression which is limiting with respect to the corresponding expression for (3.3). For example,

$$B^*(q) = B(q),\quad Q_1^*(t, q, q_1^\cdot) = \lim_{n \to +\infty} Q_1(t_n + t, q, q_1^\cdot);$$

$$B^*(q) = \lim_{n \to +\infty} B(q_1, q_2^{(n)} + q_2),\quad Q_1^*(t, q, q_1^\cdot) =$$

$$\lim_{n \to +\infty} Q_1(t_n + t, q_1, q_2^{(n)} + q_2, q_1^\cdot)\ (t_n \to +\infty,\ q_2^{(n)} \to \infty)$$

The form of system (3.4) implies that its solutions lying on the set $\{q_1^\cdot = 0\}$ must satisfy the relations

$$q_2^\cdot = 0,\quad \left(\frac{\partial \Pi}{\partial q_1} + \frac{\partial \Pi}{\partial q_2} B\right)^* = 0$$

Therefore, when the condition

$$\left\|\frac{\partial \Pi}{\partial q_1} + \frac{\partial \Pi}{\partial q_2} B\right\| \geqslant h(\|q_1\|) \geqslant 0 \tag{3.5}$$

holds, system (3.4) contains, on the set $\{q_1^{\cdot} = 0\}$, only the solutions $q_1(t) \equiv 0$. Condition (3.5) implies that there are no positions of equilibrium outside the set $\{q_1 = 0\}$, and that this property is preserved as $t_n \to +\infty$ and $q_2^{(n)} \to \infty$.

According to Theorem 2, we have the following result.

Let us assume that 1) the function $\Pi = \Pi(t, q)$ is positive-definite in q_1; 2) condition (3.5) holds; 3) the dissipative forces are such that $Q_1^T q_1^{\cdot} \leqslant -\beta(t) h(\|q_1^{\cdot}\|)$, $\beta(t) \geqslant 0$, $\beta(t) \geqslant \beta_0 > 0$ for $t \in [t_n, t_n + \nu]$ $(t_n \to +\infty,\ t_{n+1} - t_n \leqslant r,\ \nu > 0)$. Then the position of equilibrium of (3.2) is asymptotically stable with respect to $q^{\cdot}$, q_1.

The author thanks V.V. Rumyantsev for his interest and for assessing the results.

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Translated by L.K.